Geometric Group Theory
(BMath third year, 2026)

Instructions: Total time 2+1/2 hours. Solve as many problems as you like, for a maximum score of 30. Use terminology, notation and only results as covered in the course, no need to prove such results. If you wish to use a problem given in a homework or in class, please supply its full solution.

1. Let X and Y be metric spaces. Give a proof or conterexample to the statement : If X and Y are homeomorphic then they are quasi-isometric. (5)

2. Prove that the geometric realisations of the Cayley graphs $C(\mathbb{Z}, \{1\})$ and $C(\mathbb{Z}, \{2,3\})$ are not isometric. (5)

3. (a) Consider $D_\infty :=< a, b | a^2, b^2 >$. Prove that D_∞ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, here $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.
(b) Show that $c = ab$ has infinite order in D_∞.
(c) Show that D_∞ is quasi-isometric to $\mathbb{Z}$. (5+5+5)

4. Let d denote the taxi-cab metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ on $\mathbb{R}^2$.
(a) Prove that $\mathbb{R}^2$ with its Euclidean metric is quasi-isometric to $(\mathbb{R}^2, d)$.
(b) Prove that the Cayley graph of $\mathbb{Z}^2$ with respect to any finite generating set is quasi-isometric to $\mathbb{R}^2$ with its Euclidean metric. (5+5)

5. Let $H := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subset SL_2(\mathbb{Z})$. It is a fact that
$$H = \Big\{ \begin{pmatrix} 4j+1 & 2k \\ 2l & 4m+1 \end{pmatrix} \in SL_2(\mathbb{Z}) | j, k, l, m \in \mathbb{Z} \Big\}.$$
For $n = 2, 3, \cdots$ let $p_n : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_n)$ be the homomorphism given by reduction of entries modulo n, here $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. Let $\Gamma(n)$ be the kernel of p_n.
(a) Show that H has finite index in $\Gamma(2)$.
(b) Deduce that $SL_2(\mathbb{Z})$ is quasi-isometric to a tree (connected graph without cycles). (5+10)